7 (a) Let $U = \left\{p \in \mathcal{P}_4(\mathbf{R}) : \int_{-1}^{1} p = 0\right\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

Let $U := \{ p \in P_4 \mid \int_{-1}^{1} p = 0 \}$

Let $P_1 = x, \Rightarrow \int_{-1}^{1} \frac{x^2}{2} dx = \frac{1}{2} - \frac{1}{2} = 0$

$P_2 = x^2 - \frac{1}{3} \Rightarrow \int x^2 - \frac{1}{3} dx = x^3 - \frac{1}{3} x \Big|_{-1}^{1}$

$= \left(\frac{1}{3} - \frac{1}{3}\right) - \left(\frac{-1}{3} + \frac{1}{3}\right) = 0$

$P_3 = x^3 \Rightarrow \int_{-1}^{1} x^3 dx = \frac{1}{4} x^4 \Big|_{-1}^{1} = 0$

$P_4 = x^4 - \frac{1}{5} \Rightarrow \int_{-1}^{1} x^4 - \frac{1}{5} dx = \frac{x^5}{5} - \frac{x}{5} \Big|_{-1}^{1}$

$= \left(\frac{1}{5} - \frac{1}{5}\right) - \left(-\frac{1}{5} + \frac{1}{5}\right)$

$= 0$

Therfore, $P_1, P_2, P_3, P_4 \in U$

Claim1: P_1, P_2, P_3, P_4 are linearly independent.

Suppose that $\exists\, a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that

$$a_1 x + a_2\left(x^2 - \frac{1}{3}\right) + a_3 x^3 + a_4\left(x^4 - \frac{1}{5}\right) = 0$$

$$a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + \left(-\frac{a_2}{3} - \frac{a_4}{5}\right) = 0$$

By comparing relevent terms,

$$a_4 = a_3 = a_2 = a_1 = 0.$$

Claim2: $P_1, \dots, P_4$ are spans $\checkmark$

Let $f(x) = k_4 x^4 + k_3 x^3 + k_2 x^2 + k_1 x + k_0$

$$= k_4\left(x^4 - \frac{1}{5}\right) + k_3 x^3 + k_2\left(x^2 - \frac{1}{3}\right) + k_1 x + \left(\frac{1}{5} k_4 + \frac{1}{3} k_3 + k\right.$$

$$= k_4 P_4 + k_3 P_3 + k_2 P_2 + k_1 P_1 + \left(\frac{1}{5} k_4 + \frac{1}{3} k_3 + k_0\right)$$

$$\int_{-1}^{1} f(x) = \frac{k_4}{5}x^5 + \frac{k_3}{4}x^4 + \frac{k_2}{3}x^3 + \frac{k_1}{2}x^2 + k_0 x \Big|_{-1}^{1}$$

$$0 = \frac{2}{3}k_2 + \frac{2}{5}k_4 + 2k_0$$

$$0 = \frac{k_2}{3} + \frac{k_4}{5} + k_0 \longrightarrow *$$

Then, $f(x) = k_4 P_4 + k_3 P_3 + k_2 P_2 + k_1 P_1 + 0$

Thus, P_1, P_2, P_3, P_4 spans U.

Therfore, by claim 1 & claim 2, $P_1, \dots, P_4$ are basis of U.

Let $V := \{ p \in \mathcal{P}_4 \mid \int_{-1}^{1} p = 0 \}$

Let $p_1 = x$

$p_2 = x^2 - \frac{1}{3}$

$p_3 = x^3$

$p_4 = x^4 - \frac{1}{5}$

$p_0 = 1$

Claim 3: P_0, P_1, P_2, P_3, P_4 are linearly independent.

Suppose that $\exists a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that

$$a_0 + a_1 x + a_2\left(x^2 - \frac{1}{3}\right) + a_3 x^3 + a_4\left(x^4 - \frac{1}{5}\right) = 0$$

$$a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + \left(-\frac{a_2}{3} - \frac{a_4}{5} + a_0\right) = 0$$

By comparing relevent-terms,

$$a_4 = a_3 = a_2 = a_1 = a_0 = 0$$

Claim 4: $P_0 P_1, \dots, P_4$ are spans V

Let $f(x) = k_4 x^4 + k_3 x^3 + k_2 x^2 + k_1 x + k_0$

$$= k_4\left(x^4 - \frac{1}{5}\right) + k_3 x^3 + k_2\left(x^2 - \frac{1}{3}\right) + k_1 x + \left(\frac{1}{5}k_4 + \frac{1}{3}k_3 + k_0\right)$$

$$= k_4 P_4 + k_3 P_3 + k_2 P_2 + k_1 P_1 + \left(\frac{1}{5}k_4 + \frac{1}{3}k_3 + k_0\right) P_0$$

Therefore $P_0, \dots, P_4$ is a basis of V.

Let $U := \{ p \in \mathcal{P}_4 \mid \int_{-1}^{1} p = 0 \}$

Let $p_1 = x$

$p_2 = x^2 - \frac{1}{3}$.

c) Let $W = \operatorname{span}(\{1\}) = \{ k \mid k \in \mathbb{R} \}$

Then, $U + W = \mathcal{P}_4(\mathbb{F})$ (by part b))

Furthrur, observe that $U \cap W \neq \emptyset$.

Therfore, $U \oplus W = \mathcal{P}_4(\mathbb{F})$

8 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Suppose that $v_1, \ldots, v_k$ is linearly independent in V and $w \in V$

Let $W := \operatorname{span}(v_1 + w, \ldots, v_k + w)$

~~Let $V=$~~

$$v_i - v_k = (v_i + w) - (v_k + w) \in W$$

~~Let $V := \{v_1 - v_k, \ldots, v_{k-1} - v_k\}$~~

Claim: $v_1 - v_k, \ldots, v_{k-1} - v_k$ are linearly independent

Suppose that there exist $a_1, \ldots, a_{k-1} \in \mathbb{F}$ such that

$0 = a_1(v_1 - v_k) + a_2(v_2 - v_k) + \cdots + a_{k-1}(v_{k-1} - v_k)$

$0 = (a_1 v_1 + \cdots + a_{k-1} v_{k-1}) - (a_1 + \cdots + a_{k-1}) v_k$

Since $v_1, \ldots, v_k$ are linearly independent.

$$a_1 = a_2 = \cdots = a_{k-1} = (a_1 + \cdots + a_k) = 0$$

Thus, $v_1 - v_k, \ldots, v_{k-1} - v_k$ are linearly independent.

Hence W contains $(k-1)$ linearly independent vectors.

By 2.22, $(k-1) \leq \dim(W)$.

(length of linearly independent list) $\leq$ (length of spanning list)

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